Smooth PONS

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Abstract

PONS^{tm} is a basis which satisfies all of the fundamental properties of the Walsh functions (each element is piecewise constant, takes on only the values ± 1, and can be efficiently computed via a fast transform) plus three additional properties, which are false for the Walsh function: PONS is optimal with respect to a global uncertainty principle; all PONS elements have uniformly bounded crest factors; all PONS elements are QMF's. Ingrid Daubechies asked in 1991 whether there exists a smooth basis satisfying the global uncertainty principle property. In this paper we show how to transform any basis into another basis by applying the PONS construction, and thereby provide an affirmative answer to this question.

1 Introduction

The basis which has come to be known as PONS^{tm} was introduced in [4] to prove a global uncertainty principle conjecture of H. S. Shapiro. In addition to the basis functions satisfying all standard properties of the Walsh functions, namely:

- piecewise constant ± 1, changing sign only at multiples of $1/2^n$
- basis for L^2
- fast transform

PONS also has three fundamental properties of both mathematical and applied interest. They are:

- global uncertainty principle bounds
- uniform crest factor
- quadrature mirror filter identity

At the 1991 Oberwolfach wavelet conference, Ingrid Daubechies asked whether a smooth basis exists which also satisfies the global uncertainty property. Our purpose in this paper is to present a PONS-type construction which gives an affirmative answer to this question. We begin by describing (a slight variation of) the original PONS construction and briefly discussing its properties.

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2 Original PONS construction and properties

2.1 Global uncertainty principle

The classical uncertainty principle deals with how fast a single function in $L^2(\Re)$ and its Fourier transform can both approach 0 at ∞. Many of the numerous results in this area are summarized in [2]. For various reasons, in the late 1980's we investigated uncertainty principle inequalities which hold uniformly for large classes of functions. Employing a deep result of Kolmogorov, H. S. Shapiro proved [1] the impossibility of even

$$|f_n(x)| \leq C(1+|x|)^{-p}, \quad |\hat{f}_n(\xi)| \leq C(1+|\xi|)^{-p}$$

holding, when p is larger than $1/2$, for *all* $\{f_n\}$ in an *infinite orthonormal set.* Here and throughout the paper C denotes an absolute positive constant, not necessarily the same each time it is used.

Shapiro then conjectured that this was best possible, in the sense that one could have a *basis* for $L^2(\Re)$ which satisfies the above inequalities with $p = 1/2$. By employing the standard wavelet dilation and translation trick, it is easy to see that this is equivalent to having a basis $\{W_n\}$ for L^2 of a finite interval I (henceforth called a *Shapiro basis*) which satisfies:

$$|W_n(x)| \leq C \text{ for all } n \text{ and all } x \in I$$
$$\left|\hat{W}_n(\xi)\right| = \left|\int_I W_n(x)e^{-i\xi x}dx\right| \leq \frac{C}{\sqrt{1+|\xi|}}. \tag{1}$$

As shown in [4], these inequalities are satisfied by PONS.

2.2 Original PONS construction

It is, perhaps, a strange coincidence that the proof of Shapiro's conjecture was based upon his polynomial construction given in his master's thesis 41 years earlier [11]. Or, as Claude Rains says 70 minutes into *Casablanca*, "Well, maybe not so strange." Among the several profound and beautiful ideas in [11], the most elegant (in our opinion) is the recursive construction of what are now known as the Shapiro Polynomials:

$$\begin{aligned} P_0(z) &= Q_0(z) \equiv 1 \\ P_{n+1}(z) &= P_n(z) + z^{2^n}Q_n(z) \\ Q_{n+1}(z) &= P_n(z) - z^{2^n}Q_n(z). \end{aligned} \tag{2}$$

Here and below z is a complex number of modulus one and n is a nonnegative integer. It follows immediately from this definition that P_n and Q_n are polynomials of degree $2^n - 1$ with coefficients ± 1, and that

$$\begin{aligned} |P_{n+1}(z)|^2 + |Q_{n+1}(z)|^2 = 2(|P_n(z)|^2 + |Q_n(z)|^2), \text{ so that} \\ |P_n(z)|^2 + |Q_n(z)|^2 \equiv 2^{n+1}. \\ \text{Thus, } ||P_n||_\infty \leq \sqrt{2}||P_n||_{L^2}. \end{aligned} \tag{3}$$

We note that these polynomials are also referred to by the name "Rudin-Shapiro," as their first appearance in a journal article occured in [8] eight years after Shapiro's thesis. Moreover, Rudin's paper [8] was sent to the editor about one year after a talk given by Shapiro at a 1958 meeting of the American Mathematical Society, the abstract of which appeared in [7]. A detailed history of the Shapiro polynomials is presented in [9].

In antenna design, the ratio of the sup norm ("peak power") to the L^2 norm ("average power," or "energy") of a polynomial on the unit circle is called the *crest factor* or *peak factor* of the antenna array for which that polynomial is a model [3, 10]. Hence, the Shapiro Polynomials yield arrays with crest factors uniformly bounded by $\sqrt{2}$, *independent* of n. Thus, in a sense made more precise in [6], the energy of these polynomials is spread relatively evenly around the unit circle.

The PONS construction begins by defining, for each $n \geq 0$, the piecewise constant ± 1 functions $P_{n,1}(x)$ and $P_{n,2}(x)$ on the interval I (which, for convenience, we take to be $[0,1]$) to be the coefficients ϵ_i and δ_i of $P_n(z)$ and $Q_n(z)$ respectively. To be precise, for $0 \leq i \leq 2^n - 1$ and $\frac{i}{2^n} \leq x < \frac{i+1}{2^n}$, $P_{n,1}(x) = \epsilon_i$ and $P_{n,2}(x) = \delta_i$, and $P_{n,1}(1) = \epsilon_{2^n-1}$, $P_{n,2}(1) = \delta_{2^n-1}$. It is immediate that $P_{n,1}$ and $P_{n,2}$ are orthogonal on [0,1], i.e.,

$$\int_0^1 P_{n,1}(x) P_{n,2}(x) dx = 0.$$

For any $n \geq 0$ define a *piecewise Shapiro function* to be any piecewise constant ± 1 function on $[0,1]$:

- which can change sign only at points of the form $\frac{i}{2^n}$, $1 \leq i \leq 2^n - 1$, and
- whose *associated polynomial* (i.e., the polynomial whose coefficients are these ± 1's) has a *dual polynomial* satisfying (3)

It is shown in [4] that any uniformly (with respect to the degree of the associated polynomial) bounded piecewise constant (pieces of length $\frac{1}{2^n}$) function $f(x)$ whose associated polynomial has a uniformly (with respect to the degree) bounded crest factor, in particular any piecewise Shapiro function, must satisfy the global uncertainty principle inequality

$$\left| \int_0^1 f(x) e^{-2\pi i \xi x} dx \right| \leq \frac{C}{\sqrt{1+|\xi|}}.$$

Our aim is to employ piecewise Shapiro functions to construct a Shapiro basis.

As in [4], the main step in this construction is to produce enough piecewise Shapiro functions so that the characteristic function of any interval $[\frac{i}{2^n}, \frac{i+1}{2^n})$, $0 \leq i \leq 2^n - 1$, can be expressed as a finite linear combination of them. Since the collection of all such characteristic functions spans L^2, the only remaining concern will be orthogonality. Note, however, that for each $n > 0$, Shapiro's construction produces only 2 polynomials (hence 2 piecewise Shapiro functions), whereas we now require 2^n.

The most straightforward way to define the PONS generalization of the Shapiro construction is to observe the concatenation rule whereby the coefficients of $P_n(z)$ and $Q_n(z)$ yield those of $P_{n+1}(z)$ and $Q_{n+1}(z)$. Namely, with ϵ_i and δ_i again denoting the coefficients of P_n and Q_n respectively, the coefficients of P_{n+1} are the ϵ_i, $0 \leq i \leq 2^n - 1$, followed by (i.e., concatenated with) the δ_i, while those of Q_{n+1} are the ϵ_i concatenated with the *negatives* of the δ_i. Thus,

$$P_{n+1}(z) = \sum_{i=0}^{2^n-1} \epsilon_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \delta_i z^i,$$

$$Q_{n+1}(z) = \sum_{i=0}^{2^n-1} \epsilon_i z^i - z^{2^n} \sum_{i=0}^{2^n-1} \delta_i z^i, \text{ or}$$

$$\text{coefficients } (P_{n+1}) = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{2^n-1}, \delta_0, \delta_1, \ldots, \delta_{2^n-1}\},$$

$$\text{coefficients } (Q_{n+1}) = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{2^n-1}, -\delta_0, -\delta_1, \ldots, -\delta_{2^n-1}\}.$$

It is immediate that, if the ϵ_i,δ_i concatenations are reversed, then the resulting polynomials

$$R_{n+1}(z) = \sum_{i=0}^{2^n-1} \delta_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \epsilon_i z^i \text{ and}$$

$$S_{n+1}(z) = -\sum_{i=0}^{2^n-1} \delta_i z^i + z^{2^n} \sum_{i=0}^{2^n-1} \epsilon_i z^i$$

are dual, and the associated piecewise Shapiro functions are orthogonal. Moreover, since the piecewise Shapiro functions associated to $P_n(z)$ and $Q_n(z)$ are orthogonal, it is also clear that the piecewise Shapiro function associated to either $R_{n+1}(z)$ or $S_{n+1}(z)$ is automatically orthogonal to that associated to either $P_{n+1}(z)$ or $Q_{n+1}(z)$. Thus, each dual pair of Shapiro polynomials of length 2^n (i.e., degree $2^n - 1$) yields *four* mutually orthogonal piecewise Shapiro functions of length 2^{n+1}. Furthermore, if we begin with $P_1(z) = 1 + z$ and $Q_1(z) = 1 - z$ and repeatedly use this quadruple concatenation on every dual pair of Shapiro polynomials, for each n there results a symmetric $2^n \times 2^n$ Hadamard matrix $\mathcal{P}^{(n)}$ (which we refer to as a symmetric *PONS matrix*) whose rows form the required 2^n piecewise Shapiro functions of length 2^n. To illustrate, the first 3 such symmetric PONS matrices are

$$\mathcal{P}^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\mathcal{P}^{(2)} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}^{(3)} = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

The associated Shapiro polynomials are:

$$P_{1,1}(z) = P_1(z)$$

$$P_{1,2}(z) = Q_1(z)$$

and, given $P_{n,m}(z)$, $m = 1, 2, \ldots, 2^n$, for $j = 0, 1, 2 \ldots, 2^{n-1} - 1$ and $m = 4j + 1$ define $P_{n+1,m}(z)$, $m = 1, 2, \ldots, 2^{n+1}$ by:

$$\begin{aligned} P_{n+1,m} &= P_{n,2j+1} + z^{2^n} P_{n,2j+2} \\ P_{n+1,m+1} &= P_{n,2j+1} - z^{2^n} P_{n,2j+2} \\ P_{n+1,m+2} &= P_{n,2j+2} + z^{2^n} P_{n,2j+1} \\ P_{n+1,m+3} &= -P_{n,2j+2} + z^{2^n} P_{n,2j+1} \end{aligned} \quad (4)$$

We remark that the "slight variation" referred to in section 1 results from the fact that in [4] $P_{n+1,m+3}$ is the negative of that given above. We prefer the current definition, which also appears in [5], because of the symmetry of the resulting PONS matrices.

Even though, for each $n > 0$, we now have the required collection of 2^n mutually orthogonal piecewise Shapiro functions, so that we therefore have a basis for the set of all length 2^n digital signals, one last wrinkle is required to extend this to a Shapiro basis for $L^2[0, 1]$. This is because these functions need not be orthogonal for different values of n. As in [4], this problem is solved by simply defining $Q_{0,0}(z) = 1$, $Q_{0,1}(z) = 1 - z$, $Q_{n,m}(z) = (1 - z)P_{n,m}(z^2)$ for $n > 0$, $1 \leq m \leq 2^n$, and taking the values of our piecewise constant functions to be the coefficients of $Q_{n,m}(z)$. Since:

- all members of this infinite collection of piecewise constant ± 1 functions are mutually orthogonal

- the crest factor of any polynomial whose coefficients are the ± 1's of any member of the collection is at most 2
- the characteristic function of any interval $\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right), n \geq 0, 0 \leq i \leq 2^n - 1$, can be written as a finite linear combination of members of the collection

we have our Shapiro basis.

3 Constructing a smooth Shapiro basis

We first show how to transform any basis to another basis by applying the PONS construction. Thus, let $\{\phi_j(t)\}_{j=0}^{\infty}$ be a basis for $\mathcal{S}$, a normed linear space of functions on a finite interval, which (again for convenience) we take to be $[0,1]$. For each $r > 0$, $1 \leq j \leq 2^r$, $1 \leq n \leq 2^r$, let $p_n^{(r)}(j)$ be the ± 1 entry in the n^{th} row and j^{th} column of the $2^r \times 2^r$ symmetric PONS matrix $\mathcal{P}^{(r)}$. For each k, $2^r - 1 \leq k \leq 2(2^r - 1)$, define $W_k(t)$ by

$$W_k(t) = \frac{1}{\sqrt{2^r}} \sum_{j=1}^{2^r} p_{k-2^r+2}^{(r)}(j)\phi_{2^r-2+j}(t).$$

Alternatively, we can write this in matrix form by letting, for each $r \geq 0$,

$$\vec{W}^{(r)} = \langle W_{2^r-1}(t), W_{2^r}(t), W_{2^r+1}(t), \ldots, W_{2(2^r-1)}(t)\rangle^T$$

$$\text{and } \vec{\phi}^{(r)} = \langle \phi_{2^r-1}(t), \phi_{2^r}(t), \phi_{2^r+1}(t), \ldots, \phi_{2(2^r-1)}(t)\rangle^T,$$

so that

$$\vec{W}^{(r)} = \frac{1}{\sqrt{2^r}} \mathcal{P}^{(r)} \vec{\phi}^{(r)}.$$

For example, and to clarify the notation, for $r = 2$ we get

$$\begin{bmatrix} W_3(t) \\ W_4(t) \\ W_5(t) \\ W_6(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \\ \phi_6(t) \end{bmatrix}.$$

Clearly each $W_m(t)$, $0 \leq m < \infty$, is a finite linear combination of the appropriate $\phi_m(t)$ and conversely. In fact, since $\frac{1}{\sqrt{2^r}}\mathcal{P}^{(r)}$ is symmetric and unitary,

$$\vec{\phi}^{(r)} = \frac{1}{\sqrt{2^r}} \mathcal{P}^{(r)} \vec{W}^{(r)}.$$

Thus, the closed span of $\{W_m(t)\}_{m=0}^{\infty}$ is also $\mathcal{S}$. In addition, for each $r \geq 0$, if we define the dot product of a $2^r \times 1$ matrix $\vec{\nu} = \langle \nu_1(t), \nu_2(t), \ldots, \nu_{2^r}(t)\rangle^T$ with a 1×2^r matrix $\vec{\omega} = \langle \omega_1(t), \omega_2(t), \ldots, \omega_{2^r}(t)\rangle^T$ to be the $2^r \times 2^r$ matrix whose (i,j) entry is $\nu_i \cdot \omega_j = \int_0^1 \nu_i(t)\overline{\omega_j(t)}dt$ then, since

$$\vec{W}^{(r)} \cdot \vec{W}^{(r)^T} = \frac{1}{2^r} \mathcal{P}^{(r)} \vec{\phi}^{(r)} \cdot \vec{\phi}^{(r)^T} \mathcal{P}^{(r)^T} = \frac{1}{2^r} \mathcal{P}^{(r)} \mathcal{P}^{(r)^T} = I,$$

the entries of $\vec{W}^{(r)}$ form an orthonormal set. Moreover, for $r_1 \neq r_2$, any entry of $\vec{W}^{(r_1)}$ is orthogonal to any entry of $\vec{W}^{(r_2)}$, since the ϕ's occurring in $\vec{\phi}^{(r_1)}$ are distinct from those occurring in $\vec{\phi}^{(r_2)}$. To recap, $\{W_m(t)\}_{m=0}^{\infty}$ is an orthonormal set whose closed span is $\mathcal{S}$, i.e., $\{W_m(t)\}_{m=0}^{\infty}$ is a basis for $\mathcal{S}$. Finally, since each $W_m(t)$ is a finite linear combination of ϕ_m's, then all smoothness properties of the ϕ's are immediately carried over to the W's. Hence, our objective now becomes to choose the basis $\{\phi_j(t)\}_{j=0}^{\infty}$, with all of the ϕ's being smooth, so that the resulting (necessarily smooth) basis $\{W_m(t)\}_{m=0}^{\infty}$

satisfies the global uncertainty principle inequalities (1). Before proceeding we make the trivial observation that this cannot possibly occur universally, since applying the PONS basis transform to $\{W_m(t)\}_{m=0}^{\infty}$ simply yields back $\{\phi_j(t)\}_{j=0}^{\infty}$, which certainly need not satisfy (1). We now choose our initial basis $\{\phi_j(t)\}_{j=0}^{\infty}$ to be the standard Fourier basis $\{e^{2\pi ikt}\}_{k=-\infty}^{\infty}$, written as $\{e_k\}_{k=-\infty}^{\infty}$ (with $e_k = e^{2\pi ikt}$) to ease notation, and with the ordering:

$\phi_0(t) = e_0 = 1$ and, for $\mu \geq 1$,

$$\phi_j(t) = \begin{cases} e_{2^{\mu-1}-j-1} & , 2^\mu - 1 \leq j \leq 2^\mu + 2^{\mu-1} - 2 \\ e_{j+1-2^\mu} & , 2^\mu + 2^{\mu-1} - 1 \leq j \leq 2^{\mu+1} - 2 \end{cases} .$$

Thus, the ordering is given by

$$\{\phi_0, \phi_1, \phi_2, \dots\} = \{e_0, e_{-1}, e_1, e_{-2}, e_{-3}, e_2, e_3, e_{-4}, e_{-5}, e_{-6}, e_{-7}, \dots\}.$$

With this ordering, for any $r \geq 0$ and any k, $2^r - 1 \leq k \leq 2^{r+1} - 2$,

$$W_k(t) = \frac{1}{\sqrt{2^r}} \sum_{j=1}^{2^r} p_{k-2^r+2}^{(r)}(j)\phi_{2^r-2+j}(t)$$

$$= \frac{1}{\sqrt{2^r}} \sum_{j=1}^{2^{r-1}} p_{k-2^r+2}^{(r)}(j)e_{-2^{r-1}+1-j} + \frac{1}{\sqrt{2^r}} \sum_{j=2^{r-1}+1}^{2^r} p_{k-2^r+2}^{(r)}(j)e_{j-1} \tag{5}$$

Each of the two sums in the last expression for $W_k(t)$ is z to an integer power (hence a number of modulus 1) times an associated Shapiro polynomial, as given in (4), of length 2^{r-1}. Thus, each sum is bounded in modulus by $\sqrt{2^r}$, which immediately yields $|W_k(t)| \leq 2$ for all r, k, and t. It remains to obtain the second bound in (1), for $|\hat{W}_k(\xi)|$. A fundamental step in that direction is the following lemma, which shows that any finite section of any associated Shapiro polynomial has a uniformly bounded crest factor.

Lemma 1. *The crest factor of any finite section* $Q(z) = \sum_{j=k_1}^{k_2} p(j)z^j$ $(k_1 < k_2 < 2^r)$ *of any PONS polynomial* $P(z) = \sum_{j=0}^{2^r-1} p(j)z^j$, *where* $\{p(j)\}_{j=k_1}^{k_2}$ *is the consecutive set of entries of index* k_1 *through* k_2 *of any row of any* $\mathcal{P}^{(r)}$, *is less than 5.*

Proof. First note that, if any row of $\mathcal{P}^{(r)}$, $r > 0$, is split in the middle, then each half is $\pm$ a row of $\mathcal{P}^{(r-1)}$. For $i = 1, 2$, given the integers k_i of the lemma, let n_i be the unique integers defined by $2^{n_i} \leq k_i < 2^{n_i+1}$ and write $k_i - 2^{n_i} = \sum_{m=0}^{n_i-1} \gamma_m(i)2^m$, where each $\gamma_m(i) = 0$ *or* 1. By repeatedly applying the above splitting property and the fact that any full PONS polynomial has crest factor at most $\sqrt{2}$, it follows immediately that

$$\left| \sum_{j=0}^{k_i} p(j)z^j \right| \leq \sqrt{2}\sqrt{2^{n_i}} + \sqrt{2} \sum_{m=0}^{n_i-1} \gamma_m(i)\sqrt{2^m} \leq \sqrt{2} \sum_{m=0}^{n_i} 2^{\frac{m}{2}}$$

$$= \sqrt{2}\frac{2^{\frac{n_i+1}{2}} - 1}{\sqrt{2} - 1} < \sqrt{2} \cdot \frac{5}{2} \cdot \sqrt{2} \cdot 2^{\frac{n_i}{2}} = 5\sqrt{2^{n_i}} \ ,$$

so that any partial sum of any PONS polynomial has crest factor less than $\frac{5}{\sqrt{2}}$.

Now let $L = k_2 - k_1 + 1$ be the length of $Q(z)$. If L is not a power of 2, let $d \geq 0$ be the unique integer satisfying $2^d < L < 2^{d+1}$ and split $P(z)$ (which has length 2^r) into 2^{r-d} consecutive polynomials $P_1(z), P_2(z), \dots, P_{2^{r-d}}(z)$ of common length 2^d. Letting the *spectrum* of a polynomial be the set of indices where the coefficients with those indices are nonzero, the spectrum of $Q(z)$ has non-empty intersections with the spectra of two (and only two) of the 2^{r-d} polynomials P_μ. Obviously these two polynomials are consecutive ones, say P_ν and $P_{\nu+1}$. Write $Q(z) = S(z) + T(z)$, where $S(z)$ is that partial sum of $Q(z)$ which is a section of $P_\nu(z)$, so that (necessarily) $T(z)$ is that section of $Q(z)$ which is a partial sum of $P_{\nu+1}(z)$. By the above, the crest factor of $T(z)$ is less than $\frac{5}{\sqrt{2}}$.

Next, recall that the *reverse* of a polynomial $R(z) = \sum_{j=\alpha}^{\beta} c_j z^j$ is the same polynomial read in reverse order, i.e., ${}^tR(z) = \sum_{j=\alpha}^{\beta} c_{\alpha+\beta-j} z^j$. Also, $\|{}^tR\|_p = \|R\|_p$, $1 \le p \le \infty$. Since ${}^tS(z)$ is clearly a partial sum of ${}^tP_r(z)$, the crest factor of ${}^tS(z)$, and therefore the crest factor of $S(z)$, is also less than $\frac{5}{\sqrt{2}}$. Combining these crest factor bounds for $S(z)$ and $T(z)$ and letting $L_1 =$ length of $S(z)$ and $L_2 =$ length of $T(z)$, so that $L = L_1 + L_2$, we have

$$|Q(z)| \le |S(z)| + |T(z)| < \frac{5}{\sqrt{2}}(\sqrt{L_1} + \sqrt{L_2}) \le \frac{5}{\sqrt{2}}\sqrt{2}\sqrt{L} = 5\sqrt{L},$$

and the lemma is proven for the case L not a power of 2.

If L happens to be a power of 2, say $L = 2^d$, again split $P(z)$ into 2^{r-d} consecutive polynomials $P_\mu(z)$ of common length $2^d = L$, and distinguish two subcases according as $Q(z)$ is or is not one of the $P_\mu(z)$. If $Q(z) = P_\mu(z)$ for some μ, $1 \le \mu \le 2^{m-d}$, then the crest factor of Q is at most $\sqrt{2}$, and we are through. If $Q(z)$ is not one of the $P_\mu(z)$ then the spectrum of $Q(z)$ has non-empty intersections with the spectra of exactly two consecutive polynomials P_μ, say P_ν and $P_{\nu+1}$, and the above argument applies, with the same conclusion. □

Setting $p(j) = p^{(r)}_{k-2^r+2}(j+1)$, we see from (5) that

$$\hat{W}_k(\xi) = \int_0^1 W_k(t) e^{-2\pi i \xi t} dt$$

$$= \frac{1}{\sqrt{2^r}} \left\{ \sum_{j=1}^{2^{r-1}} p^{(r)}_{k-2^r+2}(j) \int_0^1 e^{-2\pi i(2^{r-1}-1+j+\xi)t} dt + \sum_{j=2^{r-1}+1}^{2^r} p^{(r)}_{k-2^r+2}(j) \int_0^1 e^{2\pi i(j-1-\xi)t} dt \right\}$$

$$= (e^{-2\pi i \xi} - 1) \frac{i}{2\pi\sqrt{2^r}} \left\{ \sum_{j=2^{r-1}}^{2^r-1} p(j - 2^{r-1}) \frac{1}{j+\xi} - \sum_{j=2^{r-1}}^{2^r-1} p(j) \frac{1}{j-\xi} \right\}. \tag{6}$$

If we now let α be 1 or -1, N be any positive integer, $F_{j,\alpha}(z)$ be the analytic function defined on $\mathcal{C}$ by

$$F_{j,\alpha}(z) = \begin{cases} \frac{1-e^{-2\pi i z}}{j-\alpha z} & if \;\; \alpha z \ne j \\ -2\pi i \alpha & if \;\; \alpha z = j \end{cases},$$

$\{q(j)\}_{j=N}^{2N-1}$ be N consecutive entries of any row of any $\mathcal{P}^{(r)}$ ($2^{r-1} \ge N$), and $\Phi_\alpha(N,\xi) = \sum_{j=N}^{2N-1} q(j) F_{j,\alpha}(\xi)$, we see from (6) that our required estimate for $|\hat{W}(\xi)|$ will follow immediately from the

Theorem 1.

$$\frac{1}{\sqrt{N}} |\Phi_\alpha(N,\xi)| \le \frac{C}{\sqrt{1+|\xi|}}$$

To prove this theorem it will be convenient to first prove an auxiliary estimate, which we give as a lemma.

Lemma 2.

$$|\Phi_\alpha(N,\xi)| \leq C$$

Proof of lemma 2. If ξ happens to be an integer then $F_{j,\alpha}(\xi) = 0$ unless $j = \alpha\xi$, while $F_{j,\alpha}(\alpha\xi) = -2\pi i\alpha$. Thus,

$$|\Phi_\alpha(N,\xi)| = \begin{cases} 2\pi & \text{if } N \leq |\xi| \leq 2N-1 \\ 0 & \text{if } \xi \text{ is any other integer} \end{cases}.$$

So we suppose, from now on, that ξ is not an integer. Also, we explicitly consider the case $\alpha = -1$, with the case $\alpha = 1$ being handled identically, and for notational convenience we replace ξ by $-s$. Thus, we now have

$$\Phi_\alpha(N,\xi) = \Phi(N,s) = (1-e^{2\pi i s})\sum_{j=N}^{2N-1}\frac{q(j)}{j-s},$$

and we will prove the required estimate $|\Phi(N,s)| \leq C$ by performing appropriate Abel (i.e., partial) summations on this sum. Let $a = [s] =$ integer part of s, so that $a < s < a+1$.

The simplest cases (for our Abel summation) occur for $s < N+1$ and $s > 2N-2$, as only one Abel summation will be needed. We start with the less simple case $N+1 < s < 2N-2$, which is equivalent to $N+1 \leq a \leq 2N-3$, where we will need *two* Abel summations. Note that this case does not occur if $N = 1$, but the lemma is trivial when $N = 1$.

Let $S = \sum_{j=N}^{a}\frac{q(j)}{j-s}$ and $T = \sum_{j=a+1}^{2N-1}\frac{q(j)}{j-s}$, so that $\Phi(N,s) = (1-e^{2\pi i s})(S+T)$. Defining $G(\rho) = \sum_{j=a+1}^{\rho} q(j)$ if $a+1 \leq \rho \leq 2N-1$, we perform a standard Abel summation on T to obtain:

$$T = \frac{G(2N-1)}{2N-1-s} + \sum_{j=a+1}^{2N-2}\left(\frac{1}{j-s} - \frac{1}{j+1-s}\right)G(j).$$

By lemma 1, the first term $\frac{G(2N-1)}{2N-1-s}$ has modulus $< \frac{5\sqrt{2N-1-s}}{2N-1-s} = \frac{5}{\sqrt{2N-1-s}} < 5$. To estimate the remaining sum, observe that $\left|\frac{1-e^{it}}{t}\right| \leq 1$ for all real t, and write

$$(1-e^{2\pi i s})\sum_{j=a+1}^{2N-2}\left(\frac{1}{j-s} - \frac{1}{j+1-s}\right)G(j) = \frac{(1-e^{2\pi i s})q(a+1)}{a+1-s} - \frac{(1-e^{2\pi i s})q(a+1)}{a+2-s}$$

$$+(1-e^{2\pi i s})\sum_{j=a+2}^{2N-2}\left(\frac{1}{j-s} - \frac{1}{j+1-s}\right)G(j).$$

Then

$$\left|\frac{(1-e^{2\pi i s})q(a+1)}{a+1-s}\right| = 2\pi\left|\frac{1-e^{2\pi i s(a+1-s)}}{2\pi(a+1-s)}\right| \leq 2\pi$$

and (because $a+2-s > 1$)

$$\left|\frac{(1-e^{2\pi i s})q(a+1)}{a+2-s}\right| < 2.$$

For the remaining sum we again use lemma 1, which immediately implies that for $a+2 \le j \le 2N-2$, $|G(j)| < 5\sqrt{j-(a+1)}$, together with the fact that $j+1-s > j-a$, to get

$$\left|\sum_{j=a+2}^{2N-2}\left(\frac{1}{j-s}-\frac{1}{j+1-s}\right)G(j)\right| = \left|\sum_{j=a+2}^{2N-2}\frac{1}{(j-s)(j+1-s)}G(j)\right|$$

$$< 5\sum_{j=a+2}^{2N-2}\frac{1}{\sqrt{j-(a+1)}(j-a)} < 5\left(\frac{1}{2}+\int_1^{\infty}\frac{1}{x^{\frac{3}{2}}}dx\right) = \frac{25}{2}.$$

Combining the above estimates, we have that $\left|(1-e^{2\pi is})T\right| < 10+2\pi+2+25 = 37+2\pi$.

The same upper bound is obtained for $\left|(1-e^{2\pi is})S\right|$ by performing a *backward* Abel summation on S, so that the lemma is proven for $N+1 < s < 2N-2$. For $s < N+1$ or $s > 2N-2$ the situation is simpler, as only one Abel summation is required in either case, standard (i.e., *forward*) for $s < N+1$ and backward for $s > 2N-2$. An upper bound similar to that above is obtained in either case (we leave the straightforward details to the reader), and lemma 2 is proven. □

Proof of Theorem 1. We now direct our attention to theorem 1. As in the proof of lemma 2, we explicitly work out the case $\alpha = -1$ and replace ξ by $-s$. We separate the cases $|s| \le 3N$ and $|s| > 3N$, noting that the factor 3 plays no special role – any number larger than 2 would do. If $|s| \le 3N$ then

$$\frac{1}{\sqrt{N}} \le \min\left(1, \sqrt{\frac{3}{|s|}}\right) \le \frac{2\sqrt{3}}{\sqrt{1+|s|}},$$

so that lemma 2 immediately yields $\frac{1}{\sqrt{N}}|\Phi_\alpha(N,\xi)| \le \frac{C}{\sqrt{1+|\xi|}}$ as required.

If $|s| > 3N$, then either $s > 3N$ or $s < -3N$. Since all j in the sum defining $\Phi_\alpha(N,\xi)$ lie between N and $2N$, if $s > 3N$ we have $j-s < 2N-s < -\frac{s}{3}$, so that $|j-s| > \frac{s}{3}$. It is immediate from (3) that

$$|F_{j,\alpha}(s)| = 2\left|\frac{\sin \pi s}{j-s}\right| \le \min\left(2\pi, \frac{2}{|j-s|},\right)$$

so that $|F_{j,\alpha}(s)| < \frac{6}{s}$ for $s > 3N$. Hence

$$|\Phi_\alpha(N,s)| \le \sum_{j=N}^{2N-1}|F_{j,\alpha}(s)| < \frac{6N}{s},$$

so that

$$\frac{1}{\sqrt{N}}|\Phi_\alpha(N,s)| < \frac{6\sqrt{N}}{s} \le \frac{6}{s}\sqrt{\frac{s}{3}} = \frac{2\sqrt{3}}{\sqrt{s}}.$$

Trivially, since $s > 3N > 3$, we have

$$\frac{1}{\sqrt{s}} \le \frac{2}{\sqrt{3}}\frac{1}{\sqrt{1+s}} = \frac{2}{\sqrt{3}}\frac{1}{\sqrt{1+|s|}},$$

and therefore

$$\frac{1}{\sqrt{N}}|\Phi_\alpha(N,s)| < \frac{4}{\sqrt{1+|s|}},$$

which completes the case $s > 3N$.

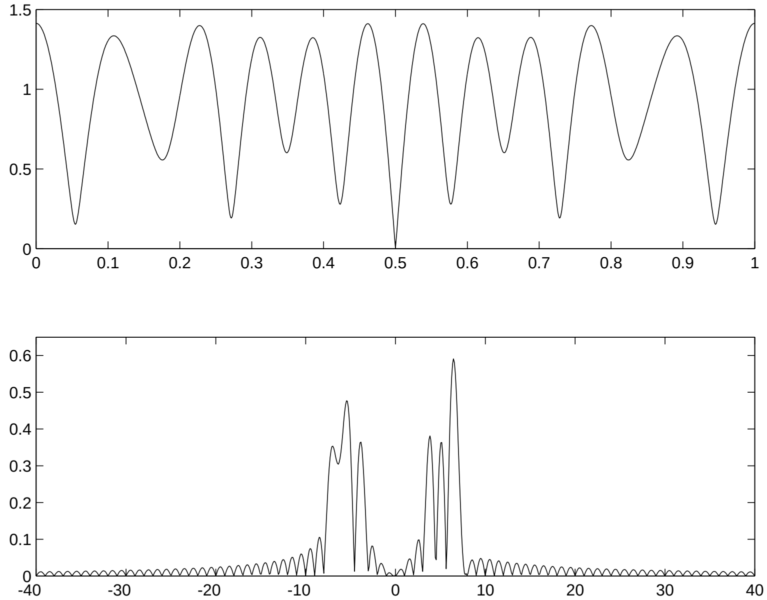

Figure 1: above, graph of $|W_7(x)|$; below, graph of $|\hat{W}_7(\xi)|$

The remaining case, $s < -3N$, is handled similarly, yielding the (slightly better) inequality

$$\frac{1}{\sqrt{N}}|\Phi_\alpha(N,s)| \leq \frac{1}{\sqrt{1+|s|}}.$$

All cases have now been taken care of and theorem 1 is proven. □

We have our smooth PONS basis. To illustrate, graphs for the typical examples $W_7(x)$ and $W_{17}(x)$, along with their Fourier transforms, are given below.

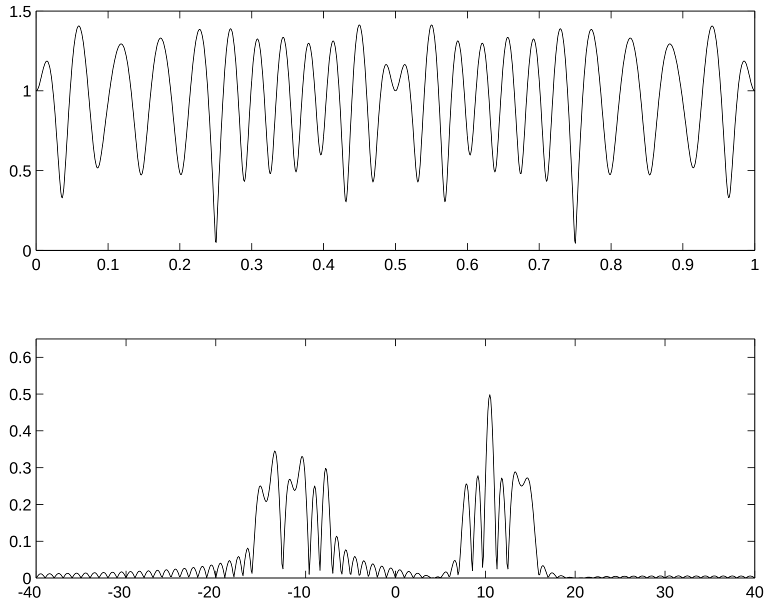

Figure 2: above, graph of $|W_{17}(x)|$; below, graph of $|\hat{W}_{17}(\xi)|$

References

[1] J. Benedetto, J.S. Byrnes, and H.S. Shapiro. Wavelet auditory models and irregular sampling. Monthly status report, February 1992.

[2] John J. Benedetto. Uncertainty principle inequalities and spectrum estimation. In J.S. Byrnes and J.L. Byrnes, editors, *Recent Advances in Fourier Analysis and its Applications*, NATO ASI. Kluwer Academic Publishers, Dordrecht, 1990.

[3] Stephen Boyd. Multitone signals with low crest factor. *IEEE Trans. Cir. & Systems*, 33:1018–1022, 1986.

[4] J.S. Byrnes. Quadrature mirror filters, low crest factor arrays, functions achieving optimal uncertainty principle bounds, and complete orthonormal sequences — a unified approach. *Applied and Computational Harmonic Analysis*, pages 261–266, 1994.

[5] J.S. Byrnes, M.A. Ramalho, G.K. Ostheimer, and I. Gertner. Discrete one dimensional signal processing method and apparatus using energy spreading coding. U.S. Patent number 5,913,186, 1999.

[6] J.S. Byrnes, B. Saffari, and H.S. Shapiro. Energy spreading and data compression using the Prometheus orthonormal set. In *Proc. 1996 IEEE Signal Processing Conf.*, Loen, Norway, 1996.

[7] H.S.Shapiro. A power series with small partial sums. *Notices of the AMS*, 6(3):366, 1958.

[8] W. Rudin. Some theorems on Fourier coefficients. *Proc. Amer. Math. Soc*, 10:855–859, 1959.

[9] B. Saffari. History of Shapiro polynomials and Golay sequences. in preparation.

[10] M.R. Schroeder. Synthesis of low peak factor signals and binary sequences with low autocorrelation. *IEEE Trans. Inf. Th.*, 16:85–89, 1970.

[11] H.S. Shapiro. Extremal problems for polynomials and power series. Sc.M. thesis, Massachusetts Institute of Technology, 1951.